A SINGLE-TYPE LOGIC FOR NATURAL LANGUAGE

KRISTINA LIEFKE†

Abstract. In this paper, we develop a single-type logic for natural language along the lines of [36]. This logic, TY_0^3, takes objects of different syntactic categories and model-theoretic domains to be structured by the same logical type. Its language, a variant of the simply-typed lambda calculus, is interpreted in partial Henkin models. We give a Gentzen-style sequent calculus for TY_0^3 and prove its soundness and completeness with respect to the class of models. To show the logic's application adequacy, we provide a TY_0^3 semantics for a standard fragment of English. Partial possible worlds, which are identified with elements in the logic's base domain, enable us to obtain the standard modal operators.

Keywords Single-type hypothesis, Data semantics, Montague grammar, Partial logic, Type theory.

1. Partee's Conjecture

Unification constitutes one of the central aims of science. More than attempting to discover large numbers of facts about the observable universe, scientists aim to establish their common underlying properties and interrelations. We strive for unification for several reasons: Far from only promoting cognitive economy and simplicity, unification explains the success of one theory (or model) in terms of another, establishes their relative consistency, and effects a mutual flow of evidential support between the two theories. Examples of relevant unified theories include electromagnetism (as a unification of electricity, magnetism, and optics), genetics (as a unification of certain effects that are produced on developing organisms), and the standard model of particle physics (as a unification of the electromagnetic, the strong, and the weak nuclear force [30]).

The present paper makes a contribution to the unificatory project. Its domain of unification constitutes the so-called linguistic, or ontological 'zoo'. The latter term was introduced in [2] to describe the plethora of objects that are assumed as the referents of certain classes (e.g. nouns, verbs, sentences) of natural language expressions. These include, but are not exhausted by, individual and plural objects (e.g. *John*, *the boys*), propositions (*John runs*), first- and higher-order properties (*runs*, *is a color*), relations (*loves*), property abstracts (*love*), kinds (*the Siberian tiger*), matter (*mud*), processes (*John running*), events (*John stumbling*), possible worlds, situations, and periods of time.

Our master objective in this paper lies in the identification of single basis for, and the description of a procedure for the bootstrapping of the above classes of objects. Richard Montague's work on formal natural language semantics [27–29] makes a significant contribution to this goal: Following Church's Simple Theory of Types [14], Montague [27] reduces the referents of the small subset of English from [29] to constructions out of two basic types of objects: individuals and propositions (or functions from indices to truth-values). From them, properties and relations are constructed as functions from individuals to propositions, functions from (functions from individuals to propositions) to propositions, and from individuals to (functions from individuals to propositions). The algebraic structure on domains enables the construction of the remaining classes of objects. However, the question remains whether it is possible to construct the ontological zoo from a single, rather than two, semantic bases. Recent research on language development [9–11, 40] seems to point in that direction.

Partee [36] takes first steps towards a complete unification of the linguistic ontology. Following Montague's method of indirect interpretation ([29]), she translates English expressions into terms of the simply-typed lambda calculus, which are then assigned values in a model. Her hypothesis ('Natural language can be modeled through the use of a single formal basis') is correspondingly formulated in terms of logical types, rather than semantic domains. Partee supports her hypothesis by sketching how a one-type system enables the formation of types for many classes of linguistic referents. However, while she identifies her basic type with the type of properties of situations, her reasons for this choice of domain, let alone her arguments for its structure, are far from imperative. This is not surprising: Partee's paper is a contribution to a set of open linguistic puzzles ([37]). Rather than attempting to formulate a sound and complete single-type logic, Partee confines herself to an indication of its semantics' main ingredients. A proof of workability is left to the scientific community.

The present paper accepts Partee's challenge. Its core objective lies in the development of a single-type logic for natural language. The paper falls into three parts, dedicated to the description of a single-type proof theory, model theory, and semantics for natural language: The following section contains an informal introduction to the objects in our single-type domains. Section 3 defines the types, terms, and models of the logic TY_0^3 with a Tarski-style truth definition and a corresponding notion of entailment. Section 4 introduces a sound Gentzen sequent calculus for TY_0^3 and proves its generalized completeness via a model existence theorem. Section 5 provides a formal definition of possible worlds. The penultimate section, 6, concerns the linguistic application of our logic: We show that TY_0^3 models a standard fragment of English. The paper closes with an assessment of the merits of single-type logic and pointers to future work.

2. Informal Analysis

To prime our intuitions, we first survey the objects in our single-type models (listed in Table 1, where $\mathcal{A}$ denotes the set of individuals). Their interrelations determine the structure of the logic TY_0^3, presented in Sections 3 through 5.

Worlds	:	Filters (and ideals) on $\mathcal{P}(\mathcal{A})$	(basic)
Individuals/Propositions	:	Filters (and ideals) on $\mathcal{P}(\mathcal{A})$	(basic)
Individual Concepts	:	Functions from worlds to individuals	(derived)
Propositional Concepts	:	Functions from worlds to propositions	(derived)
Properties	:	All functions in the domain hierarchy	(derived)

Table 1. Basic and derived single-type objects.

Following Partee's hypothesis, our single-type semantics identifies the domains of individuals and propositions. The adoption of basic worlds will be justified by the need for suitable property-representations, below.

Clearly, neither of Montague's basic objects qualifies as a single-domain candidate. Thus, while the set of individuals lacks an internal algebraic structure, propositions (or sets of indices) do not enable a suitable representation of individuals. We solve this problem through the adoption of basic sets of sets of individuals (i.e. subsets of the powerset $\mathcal{P}(\mathcal{A})$). This choice is inspired by Montague's interpretation of determiner phrases (e.g. *the man*) [29], and related work in generalized quantifier theory [5, 19, 43]. Following the latter, we represent individuals (e.g. John) by the set (e.g. $\{$is self-identical, is a man, $\dots\}$) of their associated individual-sets, or 'properties'. The partial order on property-sets allows the interpretation of linguistic connectives via the familiar set-theoretic operations. Thus, complex terms like *John and Mary* and *John or Mary* are interpreted as the intersection, respectively union of the individuals' characterizing property sets. The term *not John* is taken to denote the latters' complement in the associated algebra.

Our examples suggest the identification of individual-representations with *filters* on the set $\mathcal{P}(\mathcal{A})$. This is due to the closure of property-sets under finite intersection and entailment: By the algebraic structure on $\mathcal{P}(\mathcal{A})$, we know that, if John is characterized by the properties in the above set, he will also be characterized by the property 'is self-identical *and* is a man' (where *and* is interpreted as property-intersection, '$\cap$'). Since the set of men is, by definition, included in, e.g., the set of human beings, we further expect that the 'John'-filter, $\mathcal{F}_j = \{X \subseteq \mathcal{P}(\mathcal{A}) \,|\, \text{is a man} \subseteq X\}$, if it includes the property of being a man, will also include the property of being human such that $\mathcal{F}_j = \{X \subseteq \mathcal{P}(A) \,|\, (\text{is a man} \cap \text{is human}) \subseteq X\}$. Trivially true propositions (e.g. *John is human*) can then be represented by the intersection of the filter's set of generators and the set $\{x \in \mathcal{A} \,|\, x \in \text{is human}\}$. The existence of informative propositions is warranted by the possibility of providing proper filter extensions.[1] Thus, on the basis of the above, the proposition 'John runs' is represented by the filter $\mathcal{G}_j = \{X \subseteq \mathcal{P}(\mathcal{A}) \mid (\text{is a man} \cap \text{runs}) \subseteq X\}$, that properly includes (i.e. is more informative than) $\mathcal{F}_j$.

In single-type semantics, property attribution takes the form of filter extensions: Natural-language predicates (e.g. *runs*) are interpreted as functions from filters to filters in $\mathcal{P}(\mathcal{A})$, where a filter in the function's domain may not be more informative than the relevant filter from its range. Thus, the property 'runs' maps the 'John'-filter $\mathcal{F}_j$ to the more informative filter $\mathcal{G}_j$, such that both the argument, $\mathcal{F}_j$, and the result of functional application, $\mathcal{G}_j$, are $\mathcal{P}(\mathcal{A})$-subsets. Similarly, multi-place predicates (e.g. *loves*) are interpreted as functions from ordered n-tuples (e.g. the pair $\langle \mathcal{F}_j, \mathcal{F}_m \rangle$, with $\mathcal{F}_m$ a 'Mary'-filter) to n-tuples of filters (e.g. the pair $\langle \mathcal{F}'_j, \mathcal{F}'_m \rangle$, with $\mathcal{F}'_j := \{X \subseteq \mathcal{P}(\mathcal{A}) \mid (\bigcap_{Y \in \mathcal{F}_j} Y \cap \text{loves Mary}) \subseteq X\}$ and $\mathcal{F}'_m := \{X \subseteq \mathcal{P}(\mathcal{A}) \mid (\bigcap_{Y \in \mathcal{F}_m} Y \cap \text{is loved by John}) \subseteq X\}$), where the difference between elements of the former and the latter tuple contains at most the relation in question.

Significantly, the possibility of providing proper filter extensions is conditional on the association of individual constants with *families* of filters on $\mathcal{P}(\mathcal{A})$. This is required for the successful construction of functional domains: If every individual constant could only be interpreted as a single (ultra-)filter, informative property attributions and, hence, the possibility of distinguishing different property-representing functions, would become impossible. To obtain different individual-representations, we interpret individual constants as individual concepts [8] ([17]), i.e. functions from $\mathcal{P}(\mathcal{A})$- to $\mathcal{P}(\mathcal{A})$-subsets, whose application to different filters yields distinct values. Following Carnap, we identify filters in the function's domain and range with worlds and world-specific individuals, respectively. As a result, our ground domain will contain two different sorts of objects. Of the latter, only worlds can be directly (and unambiguously) denoted by constants in our language.

[1]This possibility also motivates our identification of single-type objects with *proper* filters, rather than *ultra*filters.

Given their seeming dissimilarity, on what grounds are the domains of individuals and worlds identified? The answer is that the representations of both types of objects share more structural properties than may, at first blush, appear: Like world-specific individuals, worlds can be characterized as sets of individual-properties. Thus, a given world is well-defined through the identification of its (individual) inhabitants and the assertion of their respective properties (via sets of $\mathcal{A}$-subsets). By the definition of individual concepts, worlds will always be at least as informative as (i.e. will never be properly contained in) their associated individuals: A world will properly contain its individual if at least one of the world's characterizing properties does not contain the represented individual, and will be identical with its individual, otherwise. We illustrate the possibility of properly contained individuals by means of an example:

Example 1. Let a world w be defined by the properties 'is self-identical' and 'is a man', with 'is self-identical' = {John, Mary, the Moon}, 'is a man' = {John}, and 'is not a man' = {Mary}, (where John, Mary, and the Moon are individuals in $\mathcal{A}$). Then, since the expressions 'John', 'John is self-identical', and 'John is a man' have the same denotation in w, the 'John'-filter at w, $\mathcal{F}_j = \{X \subseteq \mathcal{P}(\mathcal{A}) \,|\, \text{is a man} \subseteq X\}$, is identical to w. In contrast, since neither the property 'is a man' nor its complement 'is not a man' are true of the Moon in w, its filter 'the Moon'$(w) = \{X \subseteq \mathcal{P}(\mathcal{A}) \,|\, X\}$ is properly contained in the 'world'-filter $\{X \subseteq \mathcal{P}(\mathcal{A}) \,|\, \text{is a man} \subseteq X\}$, such that 'the Moon'$(w) \subsetneq w$.

On the basis of the above, what is an individual's contribution to another (possibly only partially overlapping) world? Their common identification as $\mathcal{P}(\mathcal{A})$-subsets precludes an individual's traditional existence *in* a world. However, an individual's 'world'-contribution is neither captured by the intersection or union of its world-specific instantiations: While the intersection, $\{X \subseteq \mathcal{P}(\mathcal{A}) \,|\, X\}$, of an individual's instantiations across worlds contains too little, their union provides too much information for a characterization of the world in question. We solve this dilemma through the admission of 'intermediately informative' individuals: As filters, worlds in our logic are determined by all of their (proper or improper) subsets. More than being defined exclusively by the property-set in Example 1, the world w is thus also characterized by the 'moon'-filter, 'the Moon'(w), and other less-defined filters. Naturally, none of the above is isomorphic to its underlying atom. Yet, the collection of their contributions across worlds (and thus, the acquisition of total information) yields the desired property.

The preceding paragraphs have identified proper filter extensions as the 'motor' behind propositional informativeness. By the association of individuals with their specific worlds (above), informative filter extensions involve a shift from one to another, more informative world. Given the identical generation of both worlds, this is possible only if the initial world is not totally defined. We implement this condition by dropping the Boolean law of Excluded Middle in our single-type logic in favor of a weaker axiom. The described weakening of our logic prompts a general methodological note: As the attentive reader will observe, our logic deviates in several respects from its Montagovian (or Churchian) example. While these variations are subject to individual choices, they are required by the constraints of our single-type project.

We have provided a brief sketch of worlds, world-specific individuals, individual concepts, and properties. While our observation of the need for families of 'individual'-filters has prompted the introduction of worlds and individual concepts, our presentation of propositions has remained unchanged. This is easily amended: Following the characterization of propositions as (extended) individuals, we analyze world-independent propositions as 'propositional concepts'. Like individual concepts, the latter are intuitively understood as sets of their world-specific instantiations. In contrast to their associated individuals, however, world-specific propositions (if they are informative) contain exactly one additional property.

This concludes our preliminary analysis. The following section describes the logic that is associated with our single basic type of object.

3. The Logic TY_0^3

We begin by defining the type system of the logic TY_0^3. The latter is a subsystem of a partial variant of Gallin's logic TY_2 [17] ([32]), whose types are formed exclusively through the use of the type for (characteristic functions of) sets of properties $((e;t);t)$. The language of TY_0^3 is an uncurried one-type version of the simply-typed lambda calculus ([14, 15]). Terms of TY_0^3 are interpreted in the general models from Section 3.2.

3.1. **Types and Terms.** To simplify notation, and aid the conception of the type $((e;t);t)$ as a *basic* type in the logic TY_0^3, we will hereafter abbreviate '$((e;t);t)$' as 'q'. The set of types of the single-type logic TY_0^3 is then defined as follows:

Definition 1 (TY_0^3 types)**.** The set $\mathsf{Monotype}$ of TY_0^3 types is the smallest set of strings such that, for $0 \leq n \in \mathbb{N}$, if $\alpha_1, \ldots, \alpha_n \in \mathsf{Monotype}$, then $(\alpha_1 \times \cdots \times \alpha_n) \to q \in \mathsf{Monotype}$.

The logic TY_0^3 constitutes a single-type variant of Church's Simple Theory of Types.[2] For reasons of simplicity, we replace Church's unary by an n-ary functional type logic. Complex types $(\alpha_1 \times \cdots \times \alpha_n) \to q$ form monotype correlates of Church types $\alpha_1 \to \cdots \to \alpha_n \to q$, where association is to the right. Following [42], we write $(\alpha_1 \times \cdots \times \alpha_n) \to q$ as $(\alpha_1 \ldots \alpha_n; q)$ and identify the type (q) with q.

The language $\mathcal{L}$ for single-type logic is a countable set $\cup_{\alpha \in \mathsf{Monotype}} \mathcal{L}_\alpha$ of uniquely typed non-logical constants of type α. For every TY_0^3 type α, we further assume a countable set $\mathcal{V}_\alpha$ of variables, with $\cup_{\alpha \in \mathsf{Monotype}} \mathcal{V}_\alpha$ abbreviated as $\mathcal{V}$. From these basic expressions, we form complex terms inductively with the help of application, abstraction, and a number of connectives.

Definition 2 (TY_0^3 terms)**.** Let $\alpha_1, \ldots, \alpha_n, \beta \in \mathsf{Monotype}$. The set T_α of terms of type α is defined as follows:

[2] Following Gallin's convention of subscripting a logic's name by the number of its basic types (not counting the type for formulæ) [17], we call our logic 'TY_0^3'. The zero-subscript derives from the neutralization of the distinction between individual- and proposition-types. The three-superscript indicates the partiality of its models.

i. $\mathcal{L}_\alpha, \mathcal{V}_\alpha \subseteq T_\alpha$;

ii. If $A \in T_{(\beta\alpha_1\ldots\alpha_n;q)}$ and $B \in T_\beta$, then $(AB) \in T_{(\alpha_1\ldots\alpha_n;q)}$;

iii. If $A \in T_{(\alpha_1\ldots\alpha_n;q)}$ and $x \in \mathcal{V}_\beta$, then $(\lambda x.A) \in T_{(\beta\alpha_1\ldots\alpha_n;q)}$;

iv. If $A, B \in T_\alpha$, then $\neg A$, $(A \wedge B) \in T_\alpha$,

From the above connectives, the connectives $\vee$, $\rightarrow$, and $\leftrightarrow$ are standardly defined.

Because of the ill-definedness of their logical type in our single-type system, the symbol for equality ($=$), the universal and existential quantifier ($\forall$, $\exists$), the universally true ($\top$), false ($\bot$), and undefined formula ($*$), and the familiar modal operators ($\Box, \Diamond$) are not available in the logic TY_0^3. This is a predictable consequence of our restriction of the set of TY_0^3 types to members of the set Monotype, and is unproblematic. We will see in Section 6 that the adoption of meaning postulates for non-logical constants compensates for the unavailability of the mentioned connectives.

For notational convenience, we write '$A \in T_\alpha$' as 'A_α'. We adopt the usual conventions regarding binding, freedom, and closure.

We next define the semantics of the logic TY_0^3. To this aim, we first provide a definition of general models for the logic TY_0^3.

3.2. **Models.** A model for TY_0^3 consists of a TY_0^3 frame F, an interpretation function I_F, and the variable assignment g_F. A frame for TY_0^3 is a hierarchy of typed TY_0^3 domains, which is defined as follows:

Definition 3 (General TY_0^3 frames)**.** A general TY_0^3 frame is a set $F = \{D_\alpha^F \mid \alpha \in$ Monotype$\}$ of pairwise disjoint non-empty sets such that

$$D^F_{(\alpha_1\ldots\alpha_n;q)} \subseteq \{f \mid f : (D^F_{\alpha_1} \times \cdots \times D^F_{\alpha_n}) \rightarrow D^F_q\}$$

for all TY_0^3 types $\alpha_1, \ldots, \alpha_n$.

In line with our considerations from Section 2, we define D_q^F as a subset of the space $(\mathcal{A} \to \mathbf{3}) \to \mathbf{3}$, where $\mathbf{3}$ is the ordered set of the truth-combinations *true and not false* ($\mathbf{T}$), *false and not true* ($\mathbf{F}$), and *neither true nor false* ($\mathbf{N}$). Our adoption of the set $\mathbf{3}$ (rather than of the set $\{\mathbf{T}, \mathbf{F}\}$) is motivated by the wish to enable proper extensions of single-type objects.

Domains $D^F_{(\alpha_1 \dots \alpha_n; q)}$ are sets of functions from the Cartesian product of domains of the type α_1 through α_n to the domain of the type q. We will hereafter drop the frame-subscript 'F' whenever this can be done without creating confusion. Our association of TY_0^3 domains with proper subsets of function spaces ensures the recursive axiomatizability of the entailment relation, and the associated completeness of TY_0^3. We prove the generalized completeness of TY_0^3 in Section 4.

Let us turn to the relation between TY_0^3 terms in the language $\mathcal{L}$ and their associated objects in a given TY_0^3 frame F. The latter is established by means of the interpretation function I_F and the variable assignment g_F. The function I_F is defined as follows:

Definition 4 (Interpretation)**.** The interpretation function $I_F : L \to F$ for a TY_0^3 language L and frame $F = \{D_\alpha \mid \alpha \in \mathsf{Monotype}\}$ assigns to each non-logical constant c_α of the type α a type-identical denotation in F such that $I_F(c_\alpha) \in D_\alpha$.

Variable assignments are analogously defined. Thus, the function $g_F : \mathcal{V} \to F$ applies to members of the set $\mathcal{V}_\alpha$ of variables of the type α to yield D_α-elements. Given an object $d \in D_\alpha$ and variables $x, y \in \mathcal{V}_\alpha$, we define $g_F[d/x]$ by letting $g_F[d/x](x) = d$ and $g_F[d/x](y) = g_F(y)$ if $x \neq y$. For brevity, we denote the set of all assignments g_F with respect to a given TY_0^3 frame F by $\mathcal{G}_F$. As for domains, we will drop the frame-subscript 'F' whenever suitable.

The denotation $I(c)$ (or $g(x)$) of a TY_0^3 term c (respectively x) is called the *object* designated by c (or x). Depending on their logical type, we distinguish three kinds of TY_0^3 objects: worlds (type q), individual or propositional concepts (type $(q; q)$), and properties (all complex types). Since we have identified the domain D_q with a proper subset of the space $(\mathcal{A} \to \mathbf{3}) \to \mathbf{3}$, many basic TY_0^3 objects are *partial* objects. The latter are defined as follows:

Definition 5 (TY_0^3 objects)**.** A partial object Q in the TY_0^3 domain D_q is a function that sends some type-$(e;t)$ property to the undefined truth-value $\mathbf{N}$. To emphasize the partiality of objects, the set of functions $\{\langle P_{\langle e,t\rangle}, \mathbf{T}\rangle \mid \langle P, \mathbf{T}\rangle \in Q\}$ (abbreviated Q^+) is sometimes dubbed the *denotation* of Q, and the set of functions $\{\langle P, \mathbf{F}\rangle \mid \langle P, \mathbf{F}\rangle \in Q\}$ (abbreviated Q^-) its *anti-denotation* [16, 32]. We identify type-q denotations and anti-denotations with filters (cf. Sect. 2) and ideals, respectively. The latter are subset- and union-closed subsets of $\mathcal{P}(\mathcal{A})$.

We call the difference $D_q \backslash (Q^+ \cup Q^-)$ the *gap* of Q; the intersection $(Q^+ \cap Q^-)$, its *glut*. A partial object is *coherent* if its glut is empty, *total* if its gap is empty, and *classical* if it is both coherent and total. By the identification of basic-type elements with proper filters and ideals, all basic-type objects in the logic TY_0^3 are coherent.

Our previous considerations have prepared the definition of general TY_0^3 models. However, before we can attend to this task, we first need to specify a way of converting n-ary functions into unary functions. This is required by the polyadic character of our type-forming rule from Definition 1, and the attendant restriction of functions in the domain $D_{(\alpha_1 \dots \alpha_n; q)}$ to n-tuples of (suitably typed) arguments. Slice functions [31, 32] allow the application of n-ary functions to a single argument.

To facilitate the definition of slice functions, we represent n-ary functions in $D_{(\alpha_1 \dots \alpha_n; q)}$ via sets of ordered $n+1$-tuples of the form $\langle d_1, \dots, d_n, b\rangle$, where $b \in D_q$ and $d_i \in D_{\alpha_i}$ for each $i \in \mathbb{N}$.

Let f be a function of the type $(\alpha_1 \dots \alpha_n; q)$ and let $1 \leq k \leq n \in \mathbb{N}$. Slice functions code n-ary functions into unary functions of a higher type as follows:

Definition 6 (Slice functions)**.** The k-th slice function of A applies to members of the set D_{α_k} to yield $n-1$-ary functions in the domain of the type $(\alpha_1 \dots \alpha_{k-1}\alpha_{k+1} \dots \alpha_n; q)$.

Thus, the application of the slice function F_A^k to a type-α_k object B fixes the k-th argument place of A to B, such that the following holds:

$$F_A^k(B) = \big\{\langle d_1, \dots, d_{k-1}, d_{k+1}, \dots, d_n, b\rangle \,|\, A\,(d_1, \dots, d_{k-1}, B, d_{k+1}, \dots, d_n) = b\big\}.$$

We will return to slice functions for the definition of world-specific objects and interpretation functions (Sect. 5).

On the basis of the above, general models for TY_0^3 are defined as follows:

Definition 7 (General TY_0^3 models)**.** A general model for TY_0^3 is a triple $M_F = \langle F, I_F, V_F\rangle$, consisting of a general TY_0^3 frame F, the interpretation function I_F, and the function $V_F : (\mathcal{G}_F \times \cup_\alpha T_\alpha) \to F$. The latter assigns to each non-logical TY_0^3 term A_α a type-appropriate interpretation in D_α such that

i. $V_F(g_F, c) := I_F(c)$ if $c \in \mathcal{L}$,

$V_F(g_F, x) := g_F(x)$ if $x \in \mathcal{V}$;

ii. $V_F(g_F, AB) := \big\{\big\langle \vec{d}, q\big\rangle \,|\, \langle V_F(g_F, B), \vec{d}, q\rangle \in V_F(g_F, A)\big\}$;

iii. $V_F(g_F, \lambda x_\beta.A) := \big\{\big\langle d, V_F(g_F[d/x], A)\big\rangle \,|\, d \in D_\beta\big\}$.

As a shorthand, we will often write $V_F(g_F, A)$ as $V(g, A)$, or $[\![A]\!]^g$.

The interpretations of application and abstraction (clauses (ii), (iii)) employ a variant of Muskens' slice functions ([31, 32]).

To interpret the TY_0^3 terms in Definition 2, clause (iv), we first need to define the semantic counterparts of the connectives $\wedge$, $\vee$, and $\neg$. By the considerations from Section 2, the latter are interpreted as the meet, join, and complement operations in a De Morgan algebra. De Morgan algebras have the following definition (cf. [1]):

Definition 8 (De Morgan algebra)**.** A bounded distributive lattice $\mathcal{B} = \langle B, \cap, \cup, -, 0, 1\rangle$ that satisfies the following axioms for $a, b, c \in B$:

i-v. The laws for distributive lattices;

vi. $1 \cap a = a$ and $0 \cup a = a$ (Top and Bottom);

vii. $-(a \cap b) = -a \cup -b$ and $-(a \cup b) = -a \cap -b$ (De Morgan);

viii. $- - a = a$ (Double Negation);

ix. $-0 = 1$ and $-1 = 0$ (Duality).

By the algebraic structure on D_q, all complex-type domains of the logic TY_0^3 form a De Morgan algebra. The following definition generalizes the operations on D_q to any TY_0^3 domain:

Definition 9 (Lifting)**.** Let $D_1, \dots, D_m$ be domains of the type $(\alpha_1 \dots \alpha_{n-1}; q)$ and let D_q be a De Morgan algebra. Operations on D_q can then be lifted on the function space $(D_i \to D_q)$ (with $D_i \in \{D_1, \dots, D_m\}$) by pointwise definition.

Let $[\![A_i]\!]^g$ be a function in the space $(D_i \to D_q)$. In the following, we represent $[\![A_i]\!]^g$ via the set $\big\{\big\langle [\![B_1]\!]^g, \dots, [\![B_n]\!]^g, [\![B_{n+1}]\!]^g \big\rangle \,|\, [\![A_i]\!]^g\big([\![B_1]\!]^g, \dots, [\![B_n]\!]^g\big) = [\![B_{n+1}]\!]^g \;\&\; [\![B_1]\!]^g \in D_{\alpha_1}, \dots, [\![B_n]\!]^g \in D_{\alpha_n}, [\![B_{n+1}]\!]^g \in D_q\big\}$, and abbreviate $[\![B_1]\!]^g, \dots, [\![B_n]\!]^g$ as $[\![\vec{B}]\!]^g$. Let $X = \big\{[\![A_1]\!]^g, \dots, [\![A_m]\!]^g\big\}$ be the set of functions defined above. De Morgan operations on members of the set X are then defined as follows. (To facilitate reference to functions, we use lambda abstraction in the metalanguage):

i.	$-[\![A_1]\!]^g$	$:=$	$\lambda\, [\![\vec{B}]\!]^g.\, -[\![A_1]\!]^g([\![\vec{B}]\!]^g)$;
ii.	$\bigcap X$	$:=$	$\lambda\, [\![\vec{B}]\!]^g.\, \bigcap_m \big\{[\![A_m]\!]^g([\![\vec{B}]\!]^g) \,|\, [\![A_m]\!]^g \in X\big\}$;
iii.	$\bigcup X$	$:=$	$\lambda\, [\![\vec{B}]\!]^g.\, \bigcup_m \big\{[\![A_m]\!]^g([\![\vec{B}]\!]^g) \,|\, [\![A_m]\!]^g \in X\big\}$;
iv.	$[\![A_1]\!]^g \subseteq [\![A_2]\!]^g$	$:=$	$\lambda\, [\![\vec{B}]\!]^g.\, [\![A_1]\!]^g([\![\vec{B}]\!]^g) \subseteq [\![A_2]\!]^g([\![\vec{B}]\!]^g)$;
v.	0	$:=$	$\lambda\, [\![\vec{B}]\!]^g.\, 0$;
vi.	1	$:=$	$\lambda\, [\![\vec{B}]\!]^g.\, 1$.

Above, $\lambda\, [\![\vec{B}]\!]^g.\, 0$ and $\lambda\, [\![\vec{B}]\!]^g.\, 1$ are constant functions on 0 and 1, respectively.

Note the partial definition of generalized top and bottom. The latter is warranted by the invalidity of the laws of Excluded Middle and of Consistency in the logic TY_0^3, and the attendant impossibility of defining the generalized top and bottom element via the familiar (Boolean) clauses $\lambda\, [\![\vec{B}]\!]^g.\, [\![\vec{B}]\!]^g \cup -[\![\vec{B}]\!]^g$, respectively $\lambda\, [\![\vec{B}]\!]^g.\, [\![\vec{B}]\!]^g \cap -[\![\vec{B}]\!]^g$.

To obtain the intersection (or union) of functions in the space $D_{(\alpha_1 \dots \alpha_n; q)}$, we again employ a relational coding. The defining expression in clause (ii) can then be understood as the intersection $\bigcap \big\{\big\langle [\![\vec{B}]\!]^g, [\![B_{n+1}]\!]^g \big\rangle \,|\, [\![A_i]\!]^g\big([\![\vec{B}]\!]^g\big) = [\![B_{n+1}]\!]^g\big\}$ of all sets of tuples that represent the relevant functions.

The proof that every domain $D_{(\alpha_1 \ldots \alpha_n;q)}$ forms a De Morgan algebra is analogous to the one in [18] (cf. [23]).

The above definitions enable the interpretation of the remaining terms from Definition 2 as follows:

Definition 10. Let $M_F = \langle F, I_F, V_F \rangle$ be a general model for TY_0^3 and let g_F be a variable assignment for F. Then, the following holds for all A and B of appropriate type:

i. $V_F(g_F, \neg A) \quad := \quad -V_F(g_F, A)$;

ii. $V_F(g_F, A \wedge B) \quad := \quad V_F(g_F, A) \cap V_F(g_F, B)$;

iii. $V_F(g_F, A \vee B) \quad := \quad V_F(g_F, A) \cup V_F(g_F, B)$,

where $\cap$, $\cup$, and $-$ are operations in a De Morgan algebra.

This completes our discussion of the interpretation of TY_0^3 terms. We next provide a Tarski-style truth definition for the logic TY_0^3, and a corresponding notion of entailment.

3.3. **Truth and Entailment.** In formal linguistic semantics, truth and falsity are typically defined for the (type-$(s;t)$) designators of propositions. Since such objects are, by definition, not available in the logic TY_0^3, we specify truth and falsity instead for their correspondents in the type $(q;q)$. The latter are associated with functions from the type-$((e;t);t)$ correspondents of partial indices i (i.e. the set of all type-(e,t) properties which characterize some individual inhabitant of i) to their subsets, whose members all characterize some type-e argument of the proposition (cf. Sect. 2). For instance, at the world w_1 from Example 1, the proposition 'Mary is not a man' is represented by the filter $\{X \subseteq \mathcal{P}(\mathcal{A}) \,|\, \text{is not a man} \subseteq X\}$.

In line with the above, we define the truth and falsity of a propositional concept at a world w with respect to the concept's definition for the argument w. Thus, the concept's designating TY_0^3 term A is true at w if there is some type-$((e;t);t)$ object q, such that $[\![A]\!]^g(w) = q$, and is false in w if some property P in every member of $[\![A]\!]^g$'s domain is incompatible with some property in w. The truth- and falsity-conditions of the remaining complex TY_0^3 terms then take their expected form.

Definition 11 (TY_0^3 truth)**.** Let $M_F = \langle F, I_F, V_F \rangle$ be a general TY_0^3 model for the frame F and let g_F be a variable assignment for F. Let $\mathbb{T} : (T_{(q;q)} \times W) \to \mathbf{3}$ be a truth-assignment for the single-type correspondents of type-$(s;t)$ formulæ and partial possible worlds $W \subseteq D_q$. Let $w \in W$ and $A, B \in D_{(q;q)}$. The truth-value $\mathbb{T}(A, w)$ of A at w under the assignment $\mathbb{T}$ is then defined as follows:

$$
\begin{array}{lll}
\text{i. } \mathbb{T}(A,w) = \mathbf{T} & \text{iff} & \text{for some } q \in D_{((e;t);t)},\ V_F(g_F, A)(w) = q, \\
\mathbb{T}(A,w) = \mathbf{F} & \text{iff} & \text{for some } P \in D_{(e;t)}, \text{ there is no } w \in W \text{ s.t.,} \\
 & & \text{for some } q \in D_{((e;t);t)},\ V_F(g_F, A)(w) = q \text{ and } P \in q, \\
\mathbb{T}(A,w) = \mathbf{N} & & \text{otherwise;}
\end{array}
$$

$$
\begin{array}{lll}
\text{ii. } \mathbb{T}(\neg A,w) = \mathbf{T} & \text{iff} & \mathbb{T}(A,w) = \mathbf{F}, \\
\mathbb{T}(\neg A,w) = \mathbf{F} & \text{iff} & \mathbb{T}(A,w) = \mathbf{T}, \\
\mathbb{T}(\neg A,w) = \mathbf{N} & & \text{otherwise;}
\end{array}
$$

$$
\begin{array}{lll}
\text{iii. } \mathbb{T}(A \wedge B,w) = \mathbf{T} & \text{iff} & \mathbb{T}(A,w) = \mathbf{T} \text{ and } \mathbb{T}(B,w) = \mathbf{T}, \\
\mathbb{T}(A \wedge B,w) = \mathbf{F} & \text{iff} & \mathbb{T}(A,w) = \mathbf{F} \ \text{ or } \ \mathbb{T}(B,w) = \mathbf{F}, \\
\mathbb{T}(A \wedge B,w) = \mathbf{N} & & \text{otherwise.}
\end{array}
$$

The above clauses define truth (**T**) and falsity (**F**) at a world via the truth-conditions of the Strong Kleene tables ([20]).

We call a type-$(q;q)$ term A *true* (or *false*) at w, given a TY_0^3 model M_F and variable assignment g_F iff $\mathbb{T}(A, w) = \mathbf{T}$ (respectively, **F**). We use the following abbreviation scheme:

Definition 12. Write

$$
\begin{array}{ll}
w \models_M A & \text{for} \quad \mathbb{T}(A,w) = \mathbf{T}; \\
w \models\!\!\!\!\!\!\;\dashv_M A & \text{for} \quad \mathbb{T}(A,w) = \mathbf{F}; \\
w \not\models_M A & \text{for} \quad \mathbb{T}(A,w) = \mathbf{N} \text{ or } \mathbf{F}; \\
w \not\dashv_M A & \text{for} \quad \mathbb{T}(A,w) = \mathbf{N} \text{ or } \mathbf{T}.
\end{array}
$$

Consequently, it holds that

$$
\begin{array}{lll}
\mathbb{T}(A,w) = \mathbf{T} & \text{iff} & w \models_M A \ \text{ and } \ w \not\dashv_M A\,; \\
\mathbb{T}(A,w) = \mathbf{F} & \text{iff} & w \dashv_M A \ \text{ and } \ w \not\models_M A\,.
\end{array}
$$

By the possibility of assigning undefined truth values (**N**), we are no longer able to identify the transmission of truth, $\models_M$, and the transmission of falsity, $=\!|_M$, with the transmission of non-falsity, $\not=\!|_M$, respectively the transmission of non-truth, $\not\models_M$. This is due to the greater strictness of the modeling relations $\models_M$ and $=\!|_M$. Thus, while the relation $w \models_M A$ (or $w =\!|_M A$) allows us to conclude that A (respectively, not-A) (with w representing the actual world), its counterpart, $w \not=\!|_M A$ (or $w \not\models_M A$) only prevents us from concluding that not-A (respectively, that A).

Entailment between type-$(q;q)$ terms is defined through the partial ordering $\subseteq$ on the set **3**. Below, we denote sets $\{\gamma \,|\, \gamma \in T_{(q;q)}\}$ and $\{\delta \,|\, \delta \in T_{(q;q)}\}$ of TY_0^3 terms by 'Γ' and 'Δ', respectively. Entailment for TY_0^3 is then defined as follows:

Definition 13 (TY_0^3 Entailment)**.** A set of TY_0^3 terms Γ entails a set of TY_0^3 terms Δ, i.e. $\Gamma \models_g \Delta$, iff, for all general TY_0^3 models M_F, assignments g_F, and worlds w,

$$\bigcap_{\gamma\in\Gamma} \mathbb{T}(\gamma, w) \subseteq \bigcup_{\delta\in\Delta} \mathbb{T}(\delta, w)\,.$$

According to Definition 13, Γ entails Δ iff the intersection of the w-specific evaluation of all terms in Γ is included in the union of the w-specific evaluation of all terms in Δ under the ordering $\subseteq$.

In the above, the subscript 'g' of the entailment relation $\models_g$ refers to the generality of TY_0^3 models (cf. Def. 3, 7) and the attendant recursive axiomatizability of the entailment relation. We call a TY_0^3 term γ *Henkin-valid*, or *g-valid*, if $\models_g \gamma$ for every TY_0^3 model M_F at every world-representation w.

By the partiality of the set of truth-combinations, **3**, the definition of TY_0^3 entailment from Definition 13 bifurcates into the following two conditions:

i. At every w at which all $\gamma \in \Gamma$ are true, some $\delta \in \Delta$ is true;

ii. At every w at which all $\delta \in \Delta$ are false, some $\gamma \in \Gamma$ is false.

Conditions (i) and (ii) closely resemble the entailment conditions from [6] and [7]. The first condition captures the transmission of truth ($\models_M$), the second the transmission of falsity ($=\!|_M$).

This completes our discussion of entailment in the logic TY_0^3.

4. Proof Theory

To enable a proof-theoretic characterization of TY_0^3 entailment, we identify the syntactic correspondent, $\Rightarrow$, of the relation $\models_g$. The latter denotes the partial ordering on the set $D_{(q;q)}$, such that $A \Rightarrow B$ is defined as $A \wedge B = A$. As a result, the connective $\Rightarrow$ will always result in a bivalent formula.

On the basis of the above, we can establish the following deduction theorem:

Theorem 1 (Deduction theorem for TY_0^3)**.** Let $\Gamma := \{\bigwedge_{\gamma \in \Gamma} \gamma\}$ and $\Delta := \{\bigvee_{\delta \in \Delta} \delta\}$ be sets of TY_0^3 terms of the type $(q;q)$. Then, Γ entails Δ iff Δ is deducible from Γ:

$$\Gamma \models_g \Delta \quad \text{iff} \quad \models_g \Gamma \Rightarrow \Delta\,.$$

Proof. The proof is standard.

We characterize the relation of TY_0^3 entailment via a Gentzen-style sequent calculus. A sequent $\Gamma \Rightarrow \Delta$ asserts the deducibility of a conclusion $\Delta := \bigvee_{m \in \mathbb{N}} \delta_m$ from a set $\Gamma := \bigwedge_{n \in \mathbb{N}} \gamma_n$ of type-$(q;q)$ terms.

The following definitions reflect the double-barreledness of the entailment relation:

Definition 14. A TY_0^3 model M_F for $\mathcal{L}$ *refutes* a sequent $\Gamma \Rightarrow \Delta$ if $\models_M \gamma$ for all $\gamma \in \Gamma$ and $\not\models_M \delta$ for all $\delta \in \Delta$ and if $=\!|_M\, \delta$ for all $\delta \in \Delta$ and $\not=\!|_M\, \gamma$ for all $\gamma \in \Gamma$. A sequent is g-valid if it is not refuted by any model. The set Γ of TY_0^3 terms entails Δ, i.e. $\Gamma \models \Delta$, if $\Gamma \Rightarrow \Delta$ is g-valid.

A sequent $\Gamma \Rightarrow \Delta$ is TY_0^3-provable, i.e. $\Gamma \vdash_{\mathrm{TY}_0^3} \Delta$, if there are finite Γ_0 and Δ_0, with $\Gamma_0 \subseteq \Gamma$, $\Delta_0 \subseteq \Delta$, such that Δ_0 (resp. $\Gamma_0' := \{\neg\gamma \,|\, \gamma \in \Gamma_0\}$) is deducible from the set Γ_0 ($\Delta_0' := \{\neg\delta \,|\, \delta \in \Delta_0\}$) such that Γ_0 (Δ_0') is either a sequent rule or follows by a rule from a term occurring earlier in the proof.

To facilitate the comparison with other calculi, Tables 2 and 3 (below) contain sequent rules only for the transmission of truth. Their duals (for the transmission of falsity) are obtained by taking the negation of all relevant terms in the sequent and reversing the direction of the arrow, such that the rules $\wedge\mathsf{L}$ and $\wedge\mathsf{R}$ are turned into their duals $\wedge\mathsf{L}'$ and $\wedge\mathsf{R}'$:

$$\frac{\Delta \Rightarrow \Gamma, NA, NB}{\Delta \Rightarrow \Gamma, N(A \wedge B)} \wedge\mathsf{L}' \qquad \frac{\Delta, NA \Rightarrow \Gamma \qquad \Delta, NB \Rightarrow \Gamma}{\Delta, N(A \wedge B) \Rightarrow \Gamma} \wedge\mathsf{R}',$$

with NA, NB, and $N(A \wedge B)$ signed TY_0^3 terms.

The seeming violation of the subformula property (neither NA nor NB is a subterm of $N(A \wedge B)$) is prevented by the replacement of the negation symbol $\neg$ with the sign N (reminiscent of the Polish notation for negation). The duals of all remaining rules are analogously obtained. Their generation is made explicit in Langholm's 'quadrant' sequents ([24], cf. [33]), that contain four, rather than the familiar two, structural positions.

For convenience, we omit the 'TY_0^3'-subscript of the provability relation, brackets '{', '}', and the $\emptyset$-symbol in the empty sequent. Our treatment of sequents as sets (rather than bags or lists) of terms obviates the introduction of contraction and exchange rules. We abbreviate sets of TY_0^3 terms by capital Greek letters. Let the terms A, B, and C be TY_0^3 constants of suitable type. Table 2 provides the sequent rules for our single-type logic.

$$\frac{}{A \Rightarrow A}\, R \qquad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta}\, \mathsf{cut}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\, W\mathsf{L} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\, W\mathsf{R}$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, B \Rightarrow \Delta}\, \lambda\mathsf{L} \qquad \frac{\Gamma \Rightarrow, \Delta, A}{\Gamma \Rightarrow, \Delta, B}\, \lambda\mathsf{R}$$

where $A =_{\beta\eta} B$ where $A =_{\beta\eta} B$

$$\frac{\neg\Gamma \Rightarrow \Delta}{\neg\Delta \Rightarrow \Gamma}\, \neg\mathsf{L} \qquad \frac{\Gamma \Rightarrow \neg\Delta}{\Delta \Rightarrow \neg\Gamma}\, \neg\mathsf{R}$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\, \wedge\mathsf{L} \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\, \wedge\mathsf{R}$$

Table 2. Sequent rules for TY_0^3.

The rules λL and λR assert the substitutability of $\beta\eta$-equivalent terms. From them, we are able to derive the usual rules, α, β, η, of lambda conversion:

$$\frac{}{\lambda x.A \Rightarrow \lambda y.A\{x := y\}}\,\alpha \qquad \frac{}{(\lambda x.A)B \Rightarrow A\{x := B\}}\,\beta \qquad \frac{}{\lambda x.Ax \Rightarrow A}\,\eta$$

if y is free for x in A — if B is free for x in A — if x is not free in A.

Given their standard definitions, the rules for disjunction, implication, and Double Negation (DNI) are easily derivable. The De Morgan laws (DM) and the rules of Non-Contradiction (NC) and Contraposition (CP) are also derivable. The latter are specified in Table 3:

$$\frac{}{A \Rightarrow \neg\neg A}\,\mathsf{DNI} \qquad \frac{}{\neg\neg A \Rightarrow A}\,\mathsf{DNE}$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg(A \wedge \neg A)}\,\mathsf{NC}_1 \qquad \frac{\Gamma \Rightarrow \Delta, \neg A}{\Gamma \Rightarrow \Delta, \neg(A \wedge \neg A)}\,\mathsf{NC}_2$$

$$\frac{\Gamma \Rightarrow \Delta, \neg A \vee \neg B}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)}\,\mathsf{DM}_1 \qquad \frac{\Gamma \Rightarrow \Delta, \neg A \wedge \neg B}{\Gamma \Rightarrow \Delta, \neg(A \vee B)}\,\mathsf{DM}_2$$

$$\frac{\Gamma \Rightarrow \Delta}{\neg\Delta \Rightarrow \neg\Gamma}\,\mathsf{CP}$$

TABLE 3. Derived rules for TY_0^3.

The sequent calculus for TY_0^3 is sound:

Theorem 2 (Soundness)**.** *For all sets* Γ, Δ *of* TY_0^3 *terms, if* $\Gamma \vdash \Delta$*, then* $\Gamma \models_g \Delta$*.*

Proof. The proof in standard.

4.1. **Completeness.** We prove the generalized completeness of TY_0^3, together with compactness and the Löwenheim-Skolem property, via a model existence theorem ([39]). Our proof closely follows the one in [33]: There, it is first shown that so-called 'Hintikka' sequents, which result from an unsuccessful attempt at constructing a Gentzen proof from the bottom up, are refutable. This fact is then used to establish the refutability of a large class of sequents. For brevity, we restrict ourselves to a sketch of the proof. For the original proof, the reader is referred to [33, 34].

To facilitate the definition of Hintikka sequents, we represent sequents as pairs, $\{\mathsf{L}\colon A, \mathsf{R}\colon A\}$, of signed TY_0^3 terms, where the signs L (for 'left') and R ('right') indicate the terms' structural position in the sequent. For Γ, Δ as above, the sequent $\Gamma \Rightarrow \Delta$ is then represented by the set $\{\mathsf{L}\colon A \mid A \in \Gamma\} \cup \{\mathsf{R}\colon A \mid A \in \Delta\}$.

We hereafter abbreviate $\Gamma \Rightarrow \Delta$ as Π. For simplicity, we omit the definition of Hintikka sequents for the transmission of falsity. Dual conditions are easily obtained via the relevant procedure, above.

Definition 15 (Hintikka sequents)**.** A sequent Π of TY_0^3 is a Hintikka sequent if one of the following holds:

i. $\{\mathsf{L}\colon A, \mathsf{R}\colon A\} \not\subseteq \Pi$ if $A \in T_{(q;q)}$;
ii. $\mathsf{L}\colon B \in \Pi \Longrightarrow \mathsf{L}\colon A \in \Pi$ if, for closed A, B, $A =_{\beta\eta} B$,
$\mathsf{R}\colon B \in \Pi \Longrightarrow \mathsf{R}\colon A \in \Pi$ if, for closed A, B, $A =_{\beta\eta} B$;
iii. $\mathsf{L}\colon \neg\Delta \in \Pi \Longrightarrow \mathsf{R}\colon \Delta \in \Pi$ for all closed $\delta \in \Delta$,
$\mathsf{R}\colon \neg\Gamma \in \Pi \Longrightarrow \mathsf{L}\colon \Gamma \in \Pi$ for all closed $\gamma \in \Gamma$;
iv. $\mathsf{L}\colon (A \wedge B) \in \Pi \Longrightarrow \mathsf{L}\colon A, B \in \Pi$ for all closed A, B of the same type,
$\mathsf{R}\colon (A \wedge B) \in \Pi \Longrightarrow \mathsf{R}\colon A \in \Pi$ or $\mathsf{R}\colon B \in \Pi$ for all closed A, B of the same type.

We call a Hintikka sequent Π *complete* if $\mathsf{L}\colon A \in \Pi$ or $\mathsf{R}\colon A \in \Pi$ for every term A of $\mathcal{L}$.

We next establish the refutability of Hintikka sequents by countable TY_0^3 models:

Lemma 1 (Hintikka)**.** *Every Hintikka sequent* Π *is refutable by a general* TY_0^3 *model. If* Π *is complete, it is refutable by a countable* TY_0^3 *model.*

Proof. Via the construction of a TY_0^3 model M refuting Π. By elementary considerations, M is countable if Π is complete. We identify the interpretation of TY_0^3 terms with their equivalence classes under equality. By induction on the number of connectives in a TY_0^3 term, we then establish that, for every A,

a. $\mathsf{L}\colon A \in \Pi \Longrightarrow\ \models_M A$;

b. $\mathsf{R}\colon A \in \Pi \Longrightarrow\ \not\models_M A$;

c. $\mathsf{L}\colon A \in \Pi \Longrightarrow\ =\!\mid_M A$;

d. $\mathsf{R}\colon A \in \Pi \Longrightarrow\ \not=\!\mid_M A$.

It follows that M refutes the Hintikka sequent Π.

Our proof of the model existence theorem for TY_0^3 is based on the notion of a provability property. The latter has the following definition:

Definition 16 (Provability property)**.** Let $\mathfrak{P}$ be a set of sequents in $\mathcal{L}$. The property $\mathfrak{P}$ is a provability property with respect to $\mathcal{L}$ if $\mathfrak{P}$ is closed under the sequent rules such that, if $\{\Pi_1, \dots, \Pi_n\} \subseteq \mathfrak{P}$ and if $\Pi_1, \dots, \Pi_n \backslash \Pi$ is a sequent rule, then $\Pi \in \mathfrak{P}$.

A provability property in $\mathcal{L}$ is sound if no $\Pi \in \mathfrak{P}$ is refuted by a general TY_0^3 model for $\mathcal{L}$.

Theorem 3 establishes that sequents which are not members of a sound provability property in an extended language can be extended to Hintikka sequents in that language, and are, thus, refutable.

Theorem 3 (Model Existence)**.** *Let $\mathcal{L}$ and $\mathcal{C}$ be languages for TY_0^3 such that $\mathcal{L} \cap \mathcal{C} = \emptyset$, where $\mathcal{C} = \cup_{\alpha \in \mathsf{Monotype}} \mathcal{C}_\alpha$ and where every set $\mathcal{C}_\alpha$ of non-logical constants of type α is countably infinite. Assume that $\mathfrak{P}$ is a sound provability property with respect to $\mathcal{L} \cup \mathcal{C}$ and that Π is a sequent in $\mathcal{L}$. If $\Pi \notin \mathfrak{P}$, then Π is refutable by a countable TY_0^3 model.*

Proof. Via the construction of a Hintikka sequent Π^* such that $\Pi \subseteq \Pi^*$. Let $\vartheta_1, \dots, \vartheta_n, \dots$ be an enumeration of all signed sentences in $\mathcal{L} \cup \mathcal{C}$, and let $\iota(\vartheta)$ denote the index which the signed sentence ϑ obtains in this enumeration. For every natural number n, we define a sequent Π by the following induction:

Let $\Pi_0 = \Pi$. We define Π_{n+1} as follows:

$$\Pi_{n+1} = \begin{cases} \Pi_n & \text{if } \Pi_n \cup \{\vartheta_n\} \in \mathfrak{P}; \\ \Pi_n \cup \{\vartheta_n\} & \text{if } \Pi_n \cup \{\vartheta_n\} \notin \mathfrak{P}. \end{cases}$$

That $\Pi_n \notin \mathfrak{P}$ for every n follows by a simple induction which uses Definition 16.

Define $\Pi^* = \cup_n \Pi_n$. We next establish that, for all finite sets $\{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\}$ and for all $k \geq max\{k_1, \ldots, k_n\}$, the following holds:

$$\{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \subseteq \Pi^* \quad \Leftrightarrow \quad \Pi_k \cup \{\vartheta_{k_1}, \ldots, \vartheta_{k_n}\} \notin \mathfrak{P}. \tag{1}$$

We then verify through the use of (1) that Π^* is a Hintikka sequent. Consequently, Π^* is refutable by a general TY_0^3 model. We show that Π^* (and, hence, Π) is refutable by a countable TY_0^3 model via a proof that Π^* is complete.

Theorem 3 has several desirable corollaries. In particular, the following holds, where Π is as above and where Σ ranges over sequents in $\mathcal{L} \cup \mathcal{C}$:

Corollary 1 (Generalized Compactness)**.** *For all* TY_0^3 *sequents* Π*, if* $M \models_g \Pi$*, then there is some finite* $\Pi_0 \subseteq \Pi$ *such that* $M \models_g \Pi_0$*.*

Proof. The set $\{\Sigma \,|\, M \models_g \Sigma_0$ for some finite $\Sigma_0 \subseteq \Sigma\}$ is a sound provability property.

Corollary 2 (Generalized Löwenheim-Skolem)**.** *For all* TY_0^3 *sequents* Π*, if* $M \not\models_g \Pi$*, then* Π *is refutable by a countable* TY_0^3 *model.*

Proof. The set $\{\Sigma \,|\, M \models_g \Sigma\}$ is a sound provability property.

Corollary 3 (Generalized Completeness)**.** *For all finite sets* Γ, Δ *of* TY_0^3 *terms, if* $\Gamma \models_g \Delta$*, then* $\Gamma \vdash \Delta$*.*

Proof. The set $\{\Sigma \,|\, \Sigma$ is provable$\}$ is a sound provability property.

A proof of cut-elimination (i.e. *If* $A, \Gamma \vdash \Delta$ *and* $\Gamma \vdash \Delta, A$*, then* $\Gamma \vdash \Delta$) is enabled through the replacement of the partial order on TY_0^3 frames by a (non-antisymmetric) preorder, and the associated invalidation of the axiom of Extensionality (cf. [38, 41]). The latter can later be added as a non-logical axiom.

5. Worlds

With the formal apparatus of our logic in place, we now turn to some applications. First, we show how certain basic-type objects enable the obtaining of individuals (or propositions), and the definition of the familiar necessity and possibility operators. The resulting modal language is then used to translate a Montague-style fragment of English.

In single-type semantics, possible worlds do double duty as the generators of particular individual- (and proposition-)representations and as tools for modal reasoning. Following their application in theories of intensionality, we use application to worlds to obtain, for every TY_0^3 term, a family of distinct type-appropriate objects. Following their application in modal logic, we use quantification over accessible worlds to yield the usual box and diamond operators. Because of the requirement of worlds for the generation of adequate TY_0^3 objects, we take the former use (discussed below) as primitive. Given the availability of worlds and accessibility relations, the usual modal operators are easily defined.

Section 2 has already emphasized the need for families of filters in $\mathcal{P}(\mathcal{A})$. To obtain sets of differently informative filters, we interpret individual constants as individual concepts, i.e. as functions of the type $(q;q)$, that carry a dedicated argument for type-q worlds. We isolate a world-specific individual $[\![A]\!]^g(w)$ (dubbed the *instantiation* of the constant $A_{(q;q)}$ at w) by fixing its world-slot by some particular world w.

Let w, M_F, and g_F be a specific type-q world, a TY_0^3 model, and a variable assignment, respectively. The instantiation of individual constants in w is then generalized to TY_0^3 terms of arbitrary type as follows:

Definition 17 (Instantiation)**.** The *instantiation* of a type-$(\alpha_1 \ldots \alpha_{n-1}\, q;q)$ term A at w in M_F under g_F is the result of applying the n-th slice function $F^n_{[\![A]\!]^g}$ of $[\![A]\!]^g$ to the world w.

The instantiation, $F^3_{[\![love]\!]^g}(w)$, of the term $love_{((q;q)\,(q;q)\,q;q)}$ at w is thus represented by the set of triples $\langle y, x, z\rangle$ (with $x, y, z \in D_{(q;q)}$) such that $z :=$ 'x *loves* y *at* w' is true of x and y at the world in question.

World-restricted interpretation functions capture the association of TY^3_0 constants with their world-specific instantiations as follows:

Definition 18 (Restricted denotation)**.** The world-restricted denotation function $I_{F,w} : \mathcal{L} \times \{w\} \to F$ assigns to each non-logical constant c of the type $(\alpha_1 \ldots \alpha_{n-1}\, q; q)$ a type-$(\alpha_1 \ldots \alpha_{n-1}; q)$ denotation at the world w such that $I_{F,w}(c) = F^n_{I_F(c)}(w)$.

Index-specific variable assignments are analogously defined. In line with our abbreviation for the set of all general assignments with respect to a TY^3_0 frame F, we denote the set of all w-specific assignments with respect to a TY^3_0 frame F by $\mathcal{G}_{F,w}$. As above, we drop the frame-subscript whenever possible. World-restricted variants of the function V_F are defined as follows:

Definition 19 (Restricted TY^3_0 interpretation)**.** The restricted function $V_{F,w}$: $\big((\mathcal{G}_{F,w} \times \cup_\alpha T_\alpha) \times \{w\}\big) \to F$ assigns to every pair of restricted variable assignments and TY^3_0 terms of the type $(\alpha_1 \ldots \alpha_{n-1}\, q; q)$ a denotation at the world w such that $V_{F,w}(x) := g_{F,w}(x)$ and $V_{F,w}(c) := I_{F,w}(c)$.

The restriction of I_F and V_F to specific worlds yields a dedicated TY^3_0 model for every world w. The latter have the following definition:

Definition 20 (Restricted TY^3_0 models)**.** A w-restricted model for TY^3_0 is a triple $M_{F,w} = \langle F, I_{F,w}, V_{F,w}\rangle$, consisting of a general TY^3_0 frame F, the world-restricted interpretation function $I_{F,w}$, and the function $V_{F,w}$.

A general model for TY^3_0 (cf. Def. 7) can thus be represented as the union, $\cup_{w\in W} M_{F,w}$, of all restricted TY^3_0 models $M_{F,w}$ (cf. Barwise's 'indexed unions' [3], and [32]). We abbreviate $V_{F,w}(A)$ as $[\![A]\!]^{w,g}$ or, when the world is fixed, as $[\![A]\!]^g$.

To enable an interpretation of the English expressions *necessarily* and *possibly* (cf. sentence (8)), we must first provide a formal definition of worlds and accessibility relations:

Let A and i be variables over (type-$(q;q)$) individual or propositional concepts and (type-q) worlds, respectively, where $[\![\, i \,]\!]^g = w$. The TY_0^3 term Ai then denotes the result of applying the individual (or possibility) concept $[\![A]\!]^g$ to the world w. Because of the membership of worlds in our logic's base domain, their behavior is already well-defined:

Theorem 4. Let $\Omega_{(q;q)}$ be the predicate of being a world, and let A, B, and i be of suitable type. Then, the following rules obtain, where where A and B do not depend on i (cf. [34]):

i. $\forall i.\Omega i \rightarrow \neg(A \wedge \neg A)\, i$;
ii. $\forall i.\Omega i \rightarrow ((Ai \vee \neg Ai) \vee \neg(Ai \vee \neg Ai))$;
iii. $\forall i.\Omega i \rightarrow (((\neg A \vee B)\, i) \leftrightarrow (\neg Ai \vee Bi))$.

Rules (i), (ii) assert the consistency and partiality of worlds; rule (iii) asserts their distribution over logical operators (cf. Def. 8.v). All rules are provable from the sequent rules for TY_0^3 (cf. Sect. 4, Tables 2, 3).

We denote the actual world, w_0, by the type-q constant k. By the definition of the predicate $\Omega_{(q;q)}$, w_0 is a world such that Ωk. To ensure that it is also the actual world, we stipulate the following (where $B \in T_{(q;q)}$):

iv. $\forall B.\, Bk \leftrightarrow B$.

The inclusion of an object $[\![A]\!]^g(w_1)$ in worlds $w_2 \subseteq \ldots \subseteq w_n$ is expressed by the term $((ai_1 \rightarrow i_2) \ldots \rightarrow i_n)$. By the definition of truth at a world (Def. 11), and in analogy with rule (iv), the following holds:

v. $\forall i_1 \forall i_2.(\Omega i_1 \wedge \Omega i_2) \rightarrow ((Ai_1 \rightarrow i_2) \leftrightarrow Ai_1)$.

Consequently, the statement of the inclusion $[\![A]\!]^g(w_1) \subseteq w_2$ is true if and only if $\mathbb{T}(A, w_1) = \mathbf{T}$.

From the syntax and semantics of worlds, the accessibility relation $R_{(q\,q;q)}$ is easily obtained: Let $\lambda i_2 \lambda i_1.\, R\, i_1 i_2$ formalize the expression 'i_2 *is accessible from* i_1'. The relation R inherits its properties from the partial order on D_q:

i. $\forall i_1.R\, i_1 i_1$ (Reflexivity);
ii. $\forall i_1 \forall i_2 \forall i_3.(R\, i_1 i_2 \wedge R\, i_2 i_3) \rightarrow R\, i_1 i_3)$ (Transitivity).

Other accessibility properties (e.g. symmetry, euclideanness) can be stipulated via non-logical axioms.

From the relation R, we yield the modal box operator by letting $[R] := \lambda p_{(q;q)}.\forall i\,((\Omega i \wedge Ri) \rightarrow pi)$ such that $[R]A$ is analyzed as $\forall i\,((\Omega i \wedge Ri) \rightarrow Ai)$. The diamond operator $\langle R \rangle := \lambda p.\neg[R]\neg p$ is obtained as the dual of $[R]$.

However, as we have mentioned in the definition of TY_0^3 terms, the universal quantifier $\forall$ is not available in our single-type logic. As a result, the English words *necessarily* and *possibly* cannot be translated into the above terms. To compensate for this shortcoming, we translate the two expressions into non-logical constants of suitable type. Their interpretation will then be constrained through (non-singly-typed) meaning postulates ([29]).

6. A Linguistic Application

Our objective in this paper has been to provide a single-type semantics for natural language. The present section presents a test for our achievement of the attempted unification: To compare the expressiveness and modeling power of our logic with that of competing multi-type systems ([14, 17, 29]), we model a standard subset of English ([29]; hereafter *PTQ-fragment*). Our project will be judged successful if the logic TY_0^3 enables a principled translation and interpretation of all expressions of the PTQ-fragment. We begin with some preliminary definitions.

A *fragment* is a specifiable subset of a natural language, that contains subsets of expressions from different syntactic categories. Significantly, complex expressions will only be included in a fragment if all of their basic constituents are included. To avoid the confusion of 'natural' (e.g. English) with formal terms, we refer to the fragment's basic and complex expressions as *words* and *phrases*, respectively. Instances of both are written in sans serif font. Table 4 (below) presents the words of the PTQ-fragment, together with their TY_0^3 types and translations. In the latter, italicized words (e.g. *john*, *run*) are understood as non-logical TY_0^3 constants. For notational convenience, we let the variables i, j and x, y, P, and Q range over worlds (type q), individual (or propositional) concepts (type $(q; q)$), properties of individual concepts (type-$((q; q)\, q; q)$), and properties of properties (type-$(((q; q)\, q; q)\, q; q)$) respectively.

Syntactic structures represent a phrase's combinatorial properties. To distinguish their different readings, we interpret phrases as sets, $\mathcal{S}$, of syntactic structures. Following [13], we identify the latter with labelled bracketings, whose base components represent words (by definition, members of $\mathcal{S}$), and whose remaining (bracketed) components, $[XY] \in \mathcal{S}$, represent the result of merging their respective constituents, $X, Y \in \mathcal{S}$. Type constraints on the words or phrases' logical translation prevent the generation of ill-formed structures.

The translation relation $\rightsquigarrow$ on syntactic structures is defined as follows:

Definition 21 (Type-driven translation)**.** The relation $\rightsquigarrow$ is the smallest relation between labelled bracketings and TY_0^3 terms that conforms to the rules T0 to T4, below. In the following, we let X, Y and A, B be expressions and TY_0^3 terms of the appropriate type:

Words	Translation	Type
John, Mary, Bill, ...	*john*, *mary*, *bill*...	$(q;q)$
man, woman, unicorn	*man*, *woman*, *unicorn*...	$((q;q)\,q;q)$
runs, walks, talks, ...	*run*, *walk*, *talk*...	$((q;q)\,q;q)$
finds, loves, ...	$\lambda Q\lambda y.Q(\lambda x.\,\mathit{find}\;xy),\ldots$	$((q;q)\,(q;q)\,q;q)$
seeks, ...	*seek*, ...	$((((q;q)\,q;q)\,q;q)\,(q;q)\,q;q)$
rapidly, allegedly, ...	*rapidly*, *allegedly*...	$(((q;q)\,q;q)\,(q;q)\,q;q)$
in, about	$\lambda Q\lambda P\lambda y.Q(\lambda x.\,\mathit{in}\;xPy),\ldots$	$((q;q)\,((q;q)\,q;q)\,(q;q)\,q;q)$
believes that, ...	*believe*, ...	$((q;q)\,(q;q)\,q;q)$
tries to, wishes to	*try*, *wish*	$(((q;q)\,q;q)\,(q;q)\,q;q)$
necessarily, possibly	*necessary*, *possible*	$(q;q)$
is	$\lambda Q\lambda y.Q(\lambda x.\,\mathit{is}\;xy)$	$((q;q)\,q;q)$
some, a	*some*	$(((q;q)\,q;q)\,((q;q)\,q;q)\,q;q)$
every	*every*	$(((q;q)\,q;q)\,((q;q)\,q;q)\,q;q)$
the	*the*	$(((q;q)\,q;q)\,((q;q)\,q;q)\,q;q)$
t_n	v_{α_n}	$\alpha \in$ Monotype

Table 4. PTQ-words, TY_0^3 translations, and types.

In the above table, t_n represents the trace of a moved constituent in a syntactic structure. Abstractions over their translations will be marked by superscripts.

(T0) $X \rightsquigarrow A$ if X is a word and A its translation. (Base Rule);

(T1) If $X \rightsquigarrow A$, then $[X] \rightsquigarrow A$. (Copying);

(T2) If $X \rightsquigarrow A$ and $Y \rightsquigarrow B$, then $[XY] \rightsquigarrow AB$ (Application);
if AB is well-formed, $[YX] \rightsquigarrow AB$ otherwise;

(T3) If $X \rightsquigarrow A$ and $Y \rightsquigarrow B$, then $[X^nY] \rightsquigarrow A(\lambda v_n.B)$
if $A(\lambda v_n.B)$ is well-formed; (Quantifying In);

(T4) If $X \rightsquigarrow A$ and A is reducible to B, then $X \rightsquigarrow B$. (Reduction).

Translation is accepted modulo logical equivalence. Thus, if $X \rightsquigarrow A$ and $A \equiv B$, then $X \rightsquigarrow B$. The idea of type-driven translation is due to [21].

Notably, in contrast to Montague's translation of the expressions from Table 4, the relation $\rightsquigarrow$ translates *all* words of the PTQ fragment (including auxiliaries and determiners) into non-logical TY_0^3 constants. This is required by the unavailability of many lower-type expressions in the logic TY_0^3, and the attendant absence of their associated connectives (cf. Sect. 3.1). To ensure that the constants from Table 4 receive their 'intuitive' interpretations, one must restrict the interpretation function I_F on TY_0^3 translations through the use of meaning postulates. The latter will be formulated in some lower-type language ([14, 17, 29]). The specification of these meaning postulates exceeds the scope of this paper. Instead, the interested reader is referred to [25].

In line with our considerations from Section 5, all translations from Table 4 carry an extra argument for worlds. For the partial modeling of natural language, objects of all TY_0^3 types are thus represented as sets of their associated world-specific instantiations. Primitive translations (e.g. *john*, *man*, *runs*) carry their world-indices as hidden variables, such that *john* and *run* abbreviate the TY_0^3 terms $\lambda i.john\, i$ and $\lambda x \lambda i.run\, x i$, respectively. Our treatment of 'attitudinal' constants (e.g. *believe*) will reinforce the need for an extra world-argument.

To demonstrate the expressive power of TY_0^3, we translate the relevant examples[3] from Montague's PTQ-paper. The logical rendering of structures (1)–(3) obtains the expected forms:

(1) [Bill walks] $walk\,(bill)$

(2) [[a man] walks] $some\,(man, walk)$

(3) [John [finds [a unicorn]]] $some\,(unicorn, \lambda x.\, find\ x\ john)$.

Notably, the syntactic structure in (3) yields the same translation as its associated phrase's alternative reading, [[a unicorn]^1 [John [finds t_1]]]. The replacement of finds by the complex verb seeks (analyzed as tries to find) ambiguates phrase (4) between the differently scoped translations from PTQ:

(4) John seeks a unicorn.

a. [John [seeks [a unicorn]]] $try\,(some\,(unicorn, \lambda x.\, find\ x\ john))$

b. [[a unicorn]^1[John [seeks t_1]]] $some\,(unicorn, \lambda v_1.try\,(find\ v_1, john))$

The application of the renderings in (4.a) and (4.b) to a specific-world variable k preserves their scopal differences. Significantly, Montagovian abstraction over worlds is also evitable for the translation of many belief reports. This is due to our interpretation of proper names (e.g. John) at specific ('belief'-)worlds, and the associated possibility of obtaining more fine-grained TY_0^3 objects. While

[3]For reasons of space, we limit ourselves to the best-known or most interesting examples. On their basis, the remaining examples are easily translated into TY_0^3 terms.

Montague's blocking of the inference to (7) involves the characterization of the logical translation of Phosphorus as the result of world-application (cf. (5.b)) and -abstraction (cf. (6)), respectively, our world-specific interpretation in (6) has the same outcome:

(5) a. [Mary [believes that [Phosphorus [is [the morning star]]]]]
$believe\,(the\,(morningstar, \lambda x.\, is\ x\ phosphorus), mary)$

b. [[the morning star]^1[Mary [believes that [Phosphorus [is t_1]]]]]
$the\,(morningstar, \lambda v_1.believe\,(is\ phosphorus\ v_1, mary))$

(6) $hesperus\,(k) = phosphorus\,(k)$

(7) [[the morning star]^1[Mary [believes that [Hesperus [is t_1]]]]]
$the\,(morningstar, \lambda v_1.believe\,(is\ hesperus\ v_1, mary))$

If the expression $hesperus_k = phosphorus_k$ is in Mary's belief world $[\![k]\!]^g$, the inference is valid. Otherwise, the inference is invalid. Puzzles about mathematical belief and perception reports (cf. [4, 32]) have an analogous solution.

The successful rendering of examples (1)–(7) confirms Partee's conjecture: Our single-type logic enables the modeling of a Montague-style fragment of English under the assumption of a single basic type. Since we have identified (partial) possible worlds with basic objects in our TY_0^3 frames, we are further able to interpret modal expressions at no extra cost:

(8) [Necessarily [[the morning star]^1 [is t_1]]]
$necessary\,(the\,(morningstar, \lambda x.\, is\ xx))$

As noted above, the 'resolution' of the constant *necessary* in (8) is only possible in a lower-type logic.

7. Conclusion

We have developed a single-type logic for natural language. The language of this logic, an uncurried one-type variant of the simply-typed lambda calculus, includes logical counterparts for all expressions in Montague's fragment of English. Its interpretation assigns every TY_0^3 term a world-specific interpretation in a De Morgan algebra. The provability relation over TY_0^3 terms formalizes the entailment relation between English words and phrases. By our association of complex-type domains with general Henkin frames, the latter is completely axiomatizable.

Our identification of basic-type objects with filters and ideals on the atom-set $\mathcal{A}$ yields a new type of partiality (similar to [22]), obtained neither through the assumption of uninterpreted TY_0^3 constants nor through the partial definition of their associated functions. Partial possible worlds, which are identified with elements in our logic's base domain, enable the approximation of every ultrafilter in $\mathcal{P}(\mathcal{A})$ by a family of differently well-defined interpretations. The latter have the expected properties and relations.

Naturally, many of the above choices are made for familiarity rather than necessity. Thus, we can replace Church's functional by a relational type theory, the De Morgan algebra by a (pseudo-complemented) Heyting algebra, and its double- by a definable or higher-powerset domain. Significantly, however, Partee's single-type requirement constrains the available options more strongly than most other logics for natural language. Thus, her supposition excludes a substitution of the typed by an untyped lambda calculus or the replacement of type theory by (a many-sorted) first-order logic. Similarly, algebraicity and modeling constraints block the adoption of a base domain of atoms or atom-sets alongside the Boolean law of excluded middle.

Our efforts have been limited to the development of a single-type semantics for 'Montague'-English. Clearly, the modeling of larger fragments of natural language (including, e.g., abstract, plural, and mass nouns) requires the introduction of further algebraic operations (denoted by, e.g., a nominalization [12], collectivization [26, 35], and grinding operator [23]). While the latter are well-defined for classical multi-type logics, their introduction into the logic TY_0^3 requires a careful inspection (and possible adaptation) of their logical behavior. The model-theoretic construction of the remaining objects from Section 1 is left for future work.

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Tables

Worlds	:	Filters (and ideals) on $\mathcal{P}(\mathcal{A})$	(basic)
Individuals/Propositions	:	Filters (and ideals) on $\mathcal{P}(\mathcal{A})$	(basic)
Individual Concepts	:	Functions from worlds to individuals	(derived)
Propositional Concepts	:	Functions from worlds to propositions	(derived)
Properties	:	All functions in the domain hierarchy	(derived)

Table 1. Basic and derived single-type objects.

$$\frac{}{A \Rightarrow A}\, R \qquad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta}\, \mathsf{cut}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\, W\mathsf{L} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\, W\mathsf{R}$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, B \Rightarrow \Delta}\, \lambda\mathsf{L} \qquad \frac{\Gamma \Rightarrow, \Delta, A}{\Gamma \Rightarrow, \Delta, B}\, \lambda\mathsf{R}$$

where $A =_{\beta\eta} B$ where $A =_{\beta\eta} B$

$$\frac{\neg\Gamma \Rightarrow \Delta}{\neg\Delta \Rightarrow \Gamma}\, \neg\mathsf{L} \qquad \frac{\Gamma \Rightarrow \neg\Delta}{\Delta \Rightarrow \neg\Gamma}\, \neg\mathsf{R}$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\, \wedge\mathsf{L} \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\, \wedge\mathsf{R}$$

TABLE 2. Sequent rules for TY_0^3.

$$\frac{}{A \Rightarrow \neg\neg A}\ \mathsf{DNI} \qquad \frac{}{\neg\neg A \Rightarrow A}\ \mathsf{DNE}$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg(A \wedge \neg A)}\ \mathsf{NC}_1 \qquad \frac{\Gamma \Rightarrow \Delta, \neg A}{\Gamma \Rightarrow \Delta, \neg(A \wedge \neg A)}\ \mathsf{NC}_2$$

$$\frac{\Gamma \Rightarrow \Delta, \neg A \vee \neg B}{\Gamma \Rightarrow \Delta, \neg(A \wedge B)}\ \mathsf{DM}_1 \qquad \frac{\Gamma \Rightarrow \Delta, \neg A \wedge \neg B}{\Gamma \Rightarrow \Delta, \neg(A \vee B)}\ \mathsf{DM}_2$$

$$\frac{\Gamma \Rightarrow \Delta}{\neg\Delta \Rightarrow \neg\Gamma}\ \mathsf{CP}$$

TABLE 3. Derived rules for TY_0^3.

WORDS	TRANSLATION	TYPE
John, Mary, Bill, ...	*john*, *mary*, *bill* ...	$(q;q)$
man, woman, unicorn	*man*, *woman*, *unicorn* ...	$((q;q)\,q;q)$
runs, walks, talks, ...	*run*, *walk*, *talk* ...	$((q;q)\,q;q)$
finds, loves, ...	$\lambda Q \lambda y. Q(\lambda x.\, \mathit{find}\; xy), \ldots$	$((q;q)\,(q;q)\,q;q)$
seeks, ...	*seek*, ...	$((((q;q)\,q;q)\,q;q)\,(q;q)\,q;q)$
rapidly, allegedly, ...	*rapidly*, *allegedly* ...	$(((q;q)\,q;q)\,(q;q)\,q;q)$
in, about	$\lambda Q \lambda P \lambda y. Q(\lambda x.\, \mathit{in}\; xPy), \ldots$	$((q;q)\,((q;q)\,q;q)\,(q;q)\,q;q)$
believes that, ...	*believe*, ...	$((q;q)\,(q;q)\,q;q)$
tries to, wishes to	*try*, *wish*	$(((q;q)\,q;q)\,(q;q)\,q;q)$
necessarily, possibly	*necessary*, *possible*	$(q;q)$
is	$\lambda Q \lambda y. Q(\lambda x.\, \mathit{is}\; xy)$	$((q;q)\,q;q)$
some, a	*some*	$(((q;q)\,q;q)\,((q;q)\,q;q)\,q;q)$
every	*every*	$(((q;q)\,q;q)\,((q;q)\,q;q)\,q;q)$
the	*the*	$(((q;q)\,q;q)\,((q;q)\,q;q)\,q;q)$
t_n	v_{α_n}	$\alpha \in$ Monotype

TABLE 4. PTQ-words, translations, and types.

In the above table, t_n represents the trace of a moved constituent in a syntactic structure. Abstractions over their translations will be marked by superscripts.

[†]CONTACT INFORMATION: CENTER FOR LOGIC AND PHILOSOPHY OF SCIENCE, TILBURG UNIVERSITY, P.O. BOX 90153, 5000 LE TILBURG, THE NETHERLANDS, K.LIEFKE@UVT.NL.